Disturbance Attenuation by a Frequency-Shaped Linear-Quadratic-Regulator Method

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A linear-quadratic-regulator method for frequency-dependent weighting matrices is applied to the design of a feedback control system such that a specified level of disturbance attenuation can be achieved while the stability of the system is ensured. Some inequalities relating the frequency-dependent weighting matrices to the return-difference matrix of the system are derived. They provide a straightforward design method for the return-difference matrix or, equivalently, the disturbance attenuation property. The result is applied to the design of an aircraft gust-alleviation system.

Introduction

AIRCRAFT control systems should be designed to alleviate the effect of gusts or external disturbances, because the vibration due to gusts significantly degrades aircraft flying qualities. The effect of disturbances can be reduced by designing the controller so that the sensitivity function of the resultant feedback control system is small over the frequency range where the power spectrum of gusts is significant.[1] This can be achieved by making the loop gain of the closed-loop system large over that frequency range. A high loop gain, however, significantly includes the stability and/or stability robustness of the closed-loop system.[2] A control system must be designed to alleviate the effect of disturbances without losing its stability. Therefore, a linear-quadratic-regulator (LQR) method with frequency-dependent weighting matrices[3,4] has been adopted for use herein. The LQR method ensures the stability of the closed-loop system, while the use of frequency-dependent weighting matrices adds more flexibility in designing the sensitivity matrix.

The problem of the LQR method with frequency-dependent weighting matrices has been studied by several authors. Gupta[4] solved the problem by state-space augmentation. Safonov et al.[3] developed some connections between the weighting matrices and the feedback properties of the resulting optimal system. Anderson and Mingori[5] discussed the robustness of the resulting system when high frequencies are weighted more heavily than low frequencies in the penalty on control effort.

The purpose of this paper is to develop a method of designing the return-difference matrix, the inverse of the sensitivity matrix, by means of adjusting the frequency-dependent weighting matrices.

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Problem Statement

A linear multivariable control system given by the following state-variable model is considered.

$$\dot{x}(t) = Ax(t) + Bu(t) + BDd(t) \tag{1a}$$

$$y(t) = Cx(t) \tag{1b}$$

where $x(t)$ is an n-dimensional state vector, $u(t)$ an m-dimensional control vector, $y(t)$ an m-dimensional output vector, and $d(t)$ a q-dimensional disturbance vector. A, B, C, and D are matrices of appropriate dimensions. B and C are of full rank. It may seem that the class of systems modeled by Eqs. (1) is rather specific. However, it will be shown by example that the equations of motion of an aircraft traveling in gusts belong to this class.

Let us denote the transfer function matrix from $u(s)$ to $x(s)$ using $G(s)$, i.e.,

$$G(s) = (sI - A)^{-1}B \tag{2}$$

The use of a state-feedback compensator is assumed

$$u(s) = -K(s)x(s) \tag{3}$$

The resulting feedback control system is illustrated in Fig. 1.

Disturbance Attenuation

It is easily seen that the output of the closed-loop system can be written as

$$y(s) = CG(s)(I + K(s)G(s))^{-1}Dd(s) \tag{4}$$

Thus, the effect of disturbance $d(s)$ on output $y(s)$ can be measured by $\bar{\sigma}[CG(s)(I+K(s)G(s))^{-1}D]$, where $\bar{\sigma}(\cdot)$ denotes the maximum singular value of a matrix. Also, we use $\underline{\sigma}(\cdot)$ for the minimum singular value in what follows. Then we have the following inequality:

$$\bar{\sigma}\left[CG(s)(I+K(s)G(s))^{-1}D\right] \le \frac{\bar{\sigma}[CG(s)]\bar{\sigma}(D)}{\underline{\sigma}[I+K(s)G(s)]} \tag{5}$$

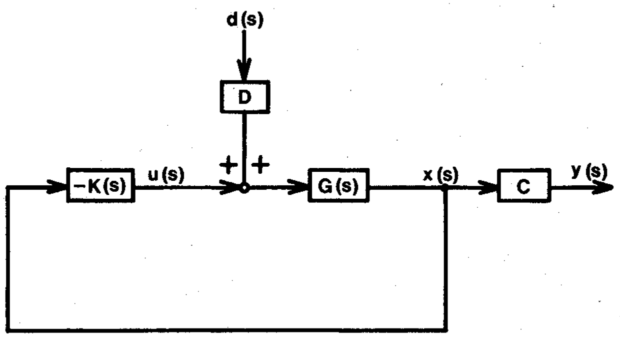

Fig. 1 Feedback control system.

Since the numerator of the right-hand side of the above inequality is independent of $K(s)$, it can be seen that the effect of disturbances can be suppressed by designing the compensator $K(s)$ such that $\underline{\sigma}[I+K(s)G(s)]$ is large over the frequency range where the power spectrum of disturbances is significant. This, however, has a tendency to decrease the stability of the resulting closed-loop system. Thus, the compensator should be designed so that the effect of disturbances is suppressed to a specified level while the stability of the closed-loop system is ensured.

For this purpose we adopt the LQR method with a frequency-shaped cost function as proposed by Gupta,[4] because the LQR method ensures stability for the closed-loop system while the frequency dependence of the cost function makes the design of $\underline{\sigma}[I+K(s)G(s)]$ easier.

We call $(I+K(s)G(s))$ the return-difference matrix, while

$$S(s)=(I+K(s)G(s))^{-1}$$

is referred to as the sensitivity matrix.[1,6]

Linear Quadratic Regulator with Frequency-Shaped Cost Function

The following cost function is considered:

$$J=\frac{1}{2}\int_{-\infty}^{\infty}[y^*(j\omega)Q(j\omega)y(j\omega)+u^*(j\omega)R(j\omega)u(j\omega)]\,d\omega \tag{6}$$

where the asterisk denotes a conjugate transpose of a vector or matrix. Both $Q(j\omega)$ and $R(j\omega)$ are $m\times m$ Hermitian positive-definite matrices, assumed to be proper rational functions of frequency squared, ω^2. Then, they can be written as follows:

$$Q(j\omega)=Q_1^*(j\omega)Q_1(j\omega) \tag{7}$$

$$R(j\omega)=R_1^*(j\omega)R_1(j\omega) \tag{8}$$

where both $Q_1(j\omega)$ and $R_1(j\omega)$ are $m\times m$ rational matrices such that

$$\lim_{\omega\to\infty} Q_1(j\omega)=D_1 \tag{9}$$

$$\lim_{\omega\to\infty} R_1(j\omega)=D_2 \tag{10}$$

are nonsingular. The problem of the linear-quadratic-regulator method with a frequency-shaped cost function is to find a control law

$$u(s)=-K(s)x(s) \tag{11}$$

that minimizes Eq. (6) subject to

$$x(s)=G(s)u(s) \tag{12}$$

Following Gupta,[4] this problem can be solved as follows. Let minimum state-space realizations of $Q_1(s)$ and $R_1(s)$ be given by (A_1,B_1,C_1,D_1) and (A_2,B_2,C_2,D_2), respectively, i.e.,

$$Q_1(s)=C_1(sI-A_1)^{-1}B_1+D_1 \tag{13}$$

$$R_1(s)=C_2(sI-A_2)^{-1}B_2+D_2 \tag{14}$$

define the following matrices:

$$A_e=\begin{bmatrix}A & 0 & 0\\ B_1C & A_1 & 0\\ 0 & 0 & A_2\end{bmatrix},\ B_e=\begin{bmatrix}B\\ 0\\ B_2\end{bmatrix},\ R_e=D_2^TD_2 \tag{15a}$$

$$Q_e=\begin{bmatrix}C^TD_1^TD_1C & C^TD_1^TC_1 & 0\\ C_1^TD_1C & C_1^TC_1 & 0\\ 0 & 0 & C_2^TC_2\end{bmatrix},\ N=\begin{bmatrix}0\\ 0\\ C_2^TD_2\end{bmatrix} \tag{15b}$$

let P be a positive-semidefinite solution to the following algebraic Riccati equation:

$$PA_e+A_e^TP-(PB_e+N)(R_e)^{-1}(PB_e+N)^T+Q_e=0 \tag{16}$$

define a matrix $K_e=[K_0,K_1,K_2]$ partitioned in correspondence with B_e, N, etc., according to

$$K_e=[K_0,K_1,K_2]=-(R_e)^{-1}(PB_e+N)^T \tag{17}$$

Then

$$K(s)=\left[I+K_2(SI-A_2)^{-1}B_2\right]^{-1}\left[K_0+K_1(sI-A_1)^{-1}B_1\right] \tag{18}$$

is a solution to the problem posed above.

The following proposition provides a sufficient condition that the Riccati equation (16) has a unique positive-semidefinite solution.

Proposition: Let $\lambda(\cdot)$ denote the set of eigenvalues of a matrix. Let A, B, C, and D be matrices such that A, CB, and D are square. Let $\rho(A,B,C,D)$ denote a set of complex numbers ρ such that

$$\begin{bmatrix}A-\rho I & B\\ C & D\end{bmatrix}$$

is nonsingular. Suppose that the following conditions hold:

Condition 1:

$$\lambda(A)\cap\lambda(A_2)\subset\mathscr{C}^-$$

Condition 2:

$$\lambda(A_1)\cap\lambda(A_2)\cap\rho(A,B,C,0)\subset\mathscr{C}^-$$

Condition 3:

$$\rho(A,B,C,0)\cap\rho(A_2,B_2,C_2,D_2)\subset\mathscr{C}^-$$

Condition 4:

$$\lambda(A)\cap\rho(A_1,B_1,C_1,D_1)\cap\rho(A_2,B_2,C_2,D_2)\subset\mathscr{C}^-$$

where $\mathscr{C}^-$ is the open left-half complex plane. Then, the Riccati equation (16) has a unique positive-semidefinite solution.

Note that every one of $CG(s)$, $Q_1(s)$, and $R_1(s)$ is a square $m \times m$ matrix; thus $\rho(A, B, C, 0)$, $\rho(A_1, B_1, C_1, D_1)$, and $\rho(A_2, B_2, C_2, D_2)$ are well defined.

A sufficient condition under which Eq. (16) has a unique solution is obtained by Anderson and Mingori[5] for a special case.

Remark: The set of conditions 1-4 is a sufficient condition that cancellation of unstable poles by unstable zeros does not occur among $[R_1(s)]^{-1}$, $CG(s)$, and $Q_1(s)$.

Problem: The problem posed herein is to design the minimum singular value of a return-difference matrix by adjusting the weighting matrices $Q(j\omega)$ and $R(j\omega)$.

Design of Return-Difference Matrix Using the LQR Method

In this section a method of designing the return-difference matrix by means of adjusting weighting matrices $Q(j\omega)$ and $R(j\omega)$ is given.

The following result is an extension of the return-difference equality given by Anderson and Mingori.[5]

Theorem 1: Let

$$G_e(s) = (sI - A_e)^{-1} B_e \tag{19}$$

Then

$$\begin{aligned} &[I + K_e G_e(j\omega)]^* R_e [I + K_e G_e(j\omega)] \\ &\quad = R(j\omega) + [CG(j\omega)]^* Q(j\omega) CG(j\omega) \end{aligned} \tag{20}$$

Proof: The proof is given in the Appendix.

The following result is immediate from Theorem 1.

Theorem 2: If $R_e = I$, the following inequalities hold.

$$\begin{aligned} &\underline{\sigma}[I + K_e G_e(j\omega)] \\ &\quad \geq [\lambda_{\min}(R(j\omega)) \\ &\quad + \lambda_{\min}[(CG(j\omega))^* Q(j\omega) CG(j\omega)]]^{1/2} \\ &\quad \geq [\lambda_{\min}(R(j\omega)) + \lambda_{\min}(Q(j\omega)) \underline{\sigma}[CG(j\omega)]]^{1/2} \end{aligned} \tag{21}$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a matrix.

Proof: The proof is given in the Appendix.

Theorem 2 provides a connection between the minimum singular value of $I + K_e G_e(j\omega)$ and the weighting matrices, while the capability of attenuating disturbances is evaluated by means of the minimum singular value of $I + K(j\omega)G(j\omega)$. The following result links these two singular values.

Theorem 3: Suppose that $R_e = I$ and $A_2 = \alpha I$ with α a nonpositive scalar and $\bar{\sigma}[C_2(j\omega I - A_2)^{-1} B_2] \leq 1$. Then

$$\begin{aligned} &\frac{\underline{\sigma}[I + K_e G_e(j\omega)]}{1 + \bar{\sigma}\left[C_2(j\omega I - A_2)^{-1} B_2\right]} \\ &\quad \leq \underline{\sigma}[I + K(j\omega) G(j\omega)] \\ &\quad \leq \frac{\underline{\sigma}[I + K_e G_e(j\omega)]}{1 - \bar{\sigma}\left[C_2(j\omega I - A_2)^{-1} B_2\right]} \end{aligned} \tag{22}$$

Proof: The proof is given in the Appendix.

When the weighting matrix $R(j\omega)$ is independent of frequency, the inequalities in Theorem 3 can be replaced by an equality. This is stated in the following corollary.

Corollary: Suppose that $R(j\omega)$ is constant. Then

$$\underline{\sigma}[I + K(j\omega) G(j\omega)] = \underline{\sigma}[I + K_e G_e(j\omega)] \tag{23}$$

Proof: The proof is obvious from Theorem 3.

A procedure of designing the minimum singular value of a return-difference matrix is described below. Let the return-difference matrix be specified by

$$\underline{\sigma}[I + K(j\omega) G(j\omega)] \geq \mu(\omega) \tag{24}$$

where $\mu(\omega)$ is a given function of ω. The proper Hermitian matrices $R(s)$ and $Q(s)$ must be found such that Eq. (24) holds. For this purpose we can set $R(j\omega) = I$. Then, it is enough to find $Q(j\omega)$ such that

$$\mu^2(\omega) I \leq I + (CG(j\omega))^* Q(j\omega) CG(j\omega) \tag{25}$$

It can be seen from Theorem 2 that Eq. (25) holds if

$$\mu^2(\omega) \leq 1 + \lambda_{\min}(Q(j\omega)) \underline{\sigma}[CG(j\omega)] \tag{26}$$

It is obvious that

$$Q(j\omega) = q(j\omega) I \tag{27}$$

is a solution to Eq. (26) if

$$q(j\omega) \geq \frac{\mu^2(\omega) - 1}{\underline{\sigma}[CG(j\omega)]} \tag{28}$$

and if unstable pole-zero cancellation does not occur between $q(s)I$ and $CG(s)$.

It should be noted that

$$\lim_{\omega \to \infty} \underline{\sigma}[CG(j\omega)] = 0$$

and that assumption (9) claims

$$\lim_{\omega \to \infty} q(j\omega) = \beta$$

where β is a positive scalar. Hence the bound $\mu^2(\omega)$ must approach 1 as fast as $\underline{\sigma}[CG(j\omega)]$ approaches zero.

When $CG(s)$ is minimum phase, i.e., $CG(s)$ has no zero and no pole in the right-half complex plane, it is easily checked that

$$Q(j\omega) = [(CG(j\omega))(CG(j\omega))^*]^{-1} q(j\omega) \tag{29}$$

is a solution to Eq. (25), where $q(j\omega)$ is a scalar function such that $1 + q(j\omega) \geq \mu^2(\omega)$. It seems that this solution is simpler to compute than Eq. (27).

Autopilot Design Example

In this section, the result of the preceding section is applied to the design of an autopilot system for aircraft longitudinal motion. The design objective is that aircraft response to vertical gusts is attenuated to a specified level while aircraft stability of motion is ensured.

The equation of longitudinal motion of an aircraft in linearized form is

$$\dot{x} = Ax + Bu + Ed, \qquad y = x$$

where $x^T = [w \quad q]$, w is the plunge velocity and q the pitch rate; $u^T = [\delta_e \quad \delta_a]$, δ_e is the elevator angle and δ_a the outboard aileron angle; and d is the vertical gust velocity. The matrices A, B, and E are given by

$$A = \begin{bmatrix} -1.099 & 251.5 \\ -0.0639 & -0.210 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.122 & -7.92 \\ -14.62 & -0.548 \end{bmatrix}$$

$$E = \begin{bmatrix} -1.099 \\ -0.0639 \end{bmatrix}$$

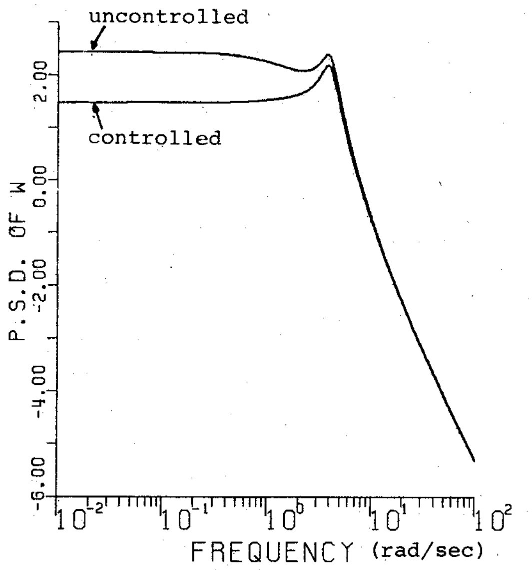

Fig. 2 Power spectral density of plunge velocity w.

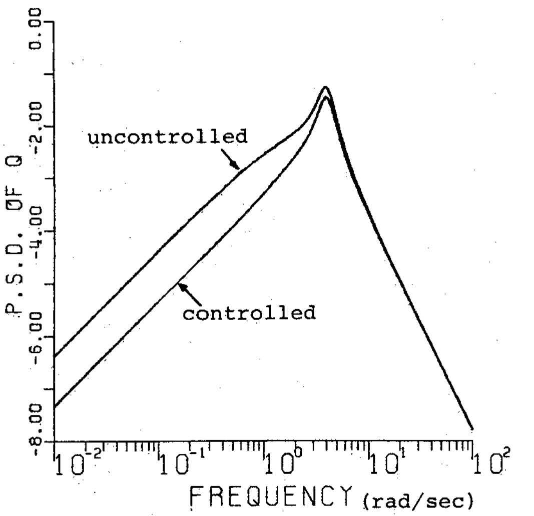

Fig. 3 Power spectral density of pitch rate q.

We remark that matrix E can be written as $E = BD$ for an appropriate matrix D; this is required as a standing assumption. The power spectral density of the gust is taken to be

$$\Phi_g = \frac{215.0}{(0.806)^2 + (0.737)^2 \omega^2}$$

This is an approximate version[7] of the well-known Dryden model. Since the corner frequency of the spectral density is about $\omega_c = 1.1$ rad/s, the authors would like to design the compensator $K(s)$ such that $\underline{\sigma}[I + K(j\omega)G(j\omega)]$ is sufficiently large in the frequency range $\omega \leq 1.1$. In view of this, let us specify

$$\underline{\sigma}[I + K(j\omega)G(j\omega)]$$
$$\geq \mu(\omega) = \left[(\omega^2 + 10.89)/(\omega^2 + 1.21)\right]^{1/2}$$

It is easily checked that the system has no zeros and $|sI - A| = s^2 + 1.309s + 16.31$. Thus, it can be seen that the system is minimum phase. Therefore, we can proceed according to Eq. (29); we choose

$$q(j\omega) = \mu^2(\omega) - 1 = \frac{9.68}{\omega^2 + 1.21}$$

then we have

$$Q(j\omega) = Q_1^*(j\omega) Q_1(j\omega)$$

where

$$Q_1(s) = \begin{bmatrix} \dfrac{0.0148s + 0.00258}{s + 1.1} & \dfrac{-0.213s - 3.75}{s + 1.1} \\ \dfrac{-0.393s - 0.431}{s + 1.1} & \dfrac{0.00327s + 98.6}{s + 1.1} \end{bmatrix}$$

The transfer function of the resulting compensator is

$$K(s) = \begin{bmatrix} \dfrac{0.0108s + 0.00182}{s + 1.1} & \dfrac{-0.151s - 2.65}{s + 1.1} \\ \dfrac{-0.277s - 0.304}{s + 1.1} & \dfrac{0.00231s + 69.7}{s + 1.1} \end{bmatrix}$$

The power spectral densities of state variables w and q of the controlled aircraft are shown in Figs. 2 and 3, respectively, where the power spectral densities of the corresponding variables of uncontrolled aircraft are also shown for comparison.

Conclusion

This paper has considered the problem of designing a control system such that the effect of disturbances can be attenuated while the stability of the closed-loop system is ensured. It is shown that a linear-quadratic-regulator method with frequency-dependent weighting matrices can be effectively applied to this problem. Some inequalities that connect those weighting matrices to the return-difference matrix of a feedback control system have been derived. By means of these inequalities, the minimum singular value of the return-difference matrix of a feedback control system can be designed quite easily.

Appendix

Proof of Proposition: It is first observed that the Riccati equation (16) can be written in the following form:

$$P(A_e - B_e R_e^{-1} N^T) + (A_e - B_e R_e^{-1} N^T)^T P$$
$$- P B_e R_e^{-1} B_e^T P + C_e^T C_e = 0 \quad \text{(A1)}$$

where

$$C_e = [D_1 C, \quad C_1, \quad 0]$$

It is known that Eq. (A1) has a unique positive-semidefinite solution P if the pair $(A_e - B_e R_e^{-1} N^T, B_e)$ is stabilizable and the pair $(C_e, A_e - B_e R_e^{-1} N^T)$ is detectable.[1,6]

Since stabilizability is not affected by state feedback,[8] to prove the stabilizability it is enough to show that (A_e, B_e) can be stabilizable. It is known[9] that (A, B) is stabilizable if, and only if, $w[\lambda I - A, B] = 0$ for some $\lambda \in \mathscr{C}^+$ implies $w = 0$, where $\mathscr{C}^+$ denotes the closed right-half complex plane.

In view of this, suppose that there exists a row vector $w = [w_1, w_2, w_3]$ such that $w[\lambda I - A_e, B_e] = 0$ for some $\lambda \in \mathscr{C}^+$. Then we have

$$w_1(\lambda I - A) - w_2 B_1 C = 0$$
$$w_2(\lambda I - A_1) = 0$$
$$w_3(\lambda I - A_2) = 0$$
$$w_1 B + w_3 B_2 = 0$$

If every one of w_1, w_2, and w_3 is not equal to zero, it follows from condition 2 that $\lambda \in \lambda(A_1) \cap \lambda(A_2) \subset \mathscr{C}^-$. This contradicts the assumption that $\lambda \in \mathscr{C}^+$, so that at least one w_i must vanish.

Thus, we assume $w_1 = 0$. It follows that $w_2[\lambda I - A_1, B_1 C] = 0$ and $w_3[\lambda I - A_2, B_2] = 0$. This claims w_2 and $w_3 = 0$ so that $w = 0$ because both (A_1, B_1) and (A_2, B_2) are controllable and C is of full rank.

Next, we assume $w_2 = 0$. If both w_1 and w_2 are not equal to zero, we have $\lambda \in \lambda(A) \cap \lambda(A_2) \subset \mathscr{C}^-$. This contradicts the assumption. Therefore, at least either w_1 or w_3 must be equal to zero. Since both (A, B) and (A_2, B_2) are controllable, it is obvious that $w_3 = 0$ if w_1 and $w_2 = 0$ and that $w_1 = 0$ if w_2 and $w_3 = 0$. Thus, we have shown that $w_2 = 0$ implies $w = 0$.

Finally, if $w_3 = 0$, we have

$$[w_1, -w_2 B_1]\begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix} = 0 \text{ and } w_2[\lambda I - A_1] = 0$$

This implies that $\lambda \in \rho(A, B, C, 0) \cap \lambda(A_1) \subset \mathscr{C}^-$. This contradicts the assumption, and hence, we must have w_1 or $w_2 = 0$. Either implies $w = 0$ as shown above.

Thus, we have shown that (A_e, B_e) is stabilizable. The property of detectability can be proved in a similar way.

Proof of Theorem 1: It follows from Eq. (16) that

$$P(j\omega I - A_e) + (-j\omega I - A_e^T)P + (PB_e + N)(R_e)^{-1}(PB_e + N)^T = Q_e \tag{A2}$$

Premultiplying and postmultiplying Eq. (A2) by $B_e^T(-j\omega I - A_e^T)^{-1}$ and $(j\omega I - A_e)^{-1}B_e$, respectively, substituting K_e for $(R_e)^{-1}(PB_e + N)^T$, and using Eq. (19), we have

$$[I + K_e G_e(j\omega)]^* R_e [I + K_e G_e(j\omega)] = R_e + G_e^*(j\omega)N + N^T G_e(j\omega) + G_e^*(j\omega)Q_e G_e(j\omega) \tag{A3}$$

It is easily verified that

$$N^T G_e(j\omega) = D_2^T C_2 (j\omega I - A_2)^{-1} B_2 \tag{A4}$$

and

$$G_e^*(j\omega)Q_e G_e(j\omega) = G^*(j\omega)Q(j\omega)G(j\omega) + \left[C_2(j\omega I - A_2)^{-1}B_2\right]^*\left[C_2(j\omega I - A_2)^{-1}B_2\right] \tag{A5}$$

Equality [Eq. (20)] follows immediately from Eqs. (A3-A5).

Proof of Theorem 2: Substituting I for R_e in Eq. (20) gives

$$[I + K_e G_e(j\omega)]^*[I + K_e G_e(j\omega)] = R(j\omega) + [CG(j\omega)]^* Q(j\omega) CG(j\omega)$$

By the definition of singular values we have

$$\underline{\sigma}[I + K_e G_e(j\omega)] = \left[\lambda_{\min}[R(j\omega) + \{CG(j\omega)\}^* Q(j\omega) CG(j\omega)]\right]^{1/2} \geq \left[\lambda_{\min}[R(j\omega)] + \lambda_{\min}[\{CG(j\omega)\}^* Q(j\omega) CG(j\omega)]\right]$$

This proves the first inequality. It remains to prove that

$$\lambda_{\min}[\{CG(j\omega)\}^* Q(j\omega) CG(j\omega)] \geq \lambda_{\min}[Q(j\omega)]\underline{\sigma}[CG(j\omega)] \tag{A6}$$

Since $Q(j\omega)$ is Hermitian, we have $Q(j\omega) - \lambda_{\min}[Q(j\omega)]I \geq 0$, so that

$$\{CG(j\omega)\}^* Q(j\omega) CG(j\omega) - \lambda_{\min}[Q(j\omega)]\{CG(j\omega)\}^* CG(j\omega) = \{CG(j\omega)\}^*\{Q(j\omega) - \lambda_{\min}[Q(j\omega)I]I\}CG(j\omega) \geq 0$$

This verifies Eq. (A6).

Proof of Theorem 3: We first show that

$$\bar{\sigma}\left[C_2(j\omega I - A_2)^{-1}B_2\right] \geq \bar{\sigma}\left[K_2(j\omega I - A_2)^{-1}B_2\right] \tag{A7}$$

Defining appropriate matrices and noting that $R_e = I$, the Riccati equation (16) can be written as follows:

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}\begin{bmatrix} \tilde{A} & 0 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} \tilde{A} & 0 \\ 0 & A_2 \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} - \left\{\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}\begin{bmatrix} \tilde{B} \\ B_2 \end{bmatrix} + \begin{bmatrix} 0 \\ C_2^T D_2 \end{bmatrix}\right\} \times \left\{\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}\begin{bmatrix} \tilde{B} \\ B_2 \end{bmatrix} + \begin{bmatrix} 0 \\ C_2^T D_2 \end{bmatrix}\right\}^T + \begin{bmatrix} \tilde{Q} & 0 \\ 0 & C_2^T C_2 \end{bmatrix} = 0 \tag{A8}$$

It is easy to see from Eq. (17) that $K_2 = (P_2^T \tilde{B} + P_3 B_2 + C_2^T D_2)^T$. Therefore, the final matrix of Eq. (A8) becomes

$$P_3(j\omega I - A_2) + (-j\omega I - A_2^T)P_3 + K_2^T K_2 - C_2^T C_2 = 0 \tag{A9}$$

Premultiplying and postmultiplying Eq. (A9) by $B_2^T(j\omega I - A_2^T)^{-1}$ and $(j\omega I - A_2)^{-1}B_2$, respectively, and rearranging, we have

$$\left[K_2(j\omega I - A_2)^{-1}B_2\right]^*\left[K_2(j\omega I - A_2)^{-1}B_2\right] = \left[C_2(j\omega I - A_2)^{-1}B_2\right]^*\left[C_2(j\omega I - A_2)^{-1}B_2\right] - \left\{\left[B_2^T P_3(j\omega I - A_2)^{-1}B_2\right]^* + \left[B_2^T P_3(j\omega I - A_2)^{-1}B_2\right]\right\} \tag{A10}$$

Noting that P_3 is positive-semidefinite and symmetric and $A_2 = \alpha I$ $(\alpha \leq 0)$, we have $P_3 A_2 + A_2^T P_3 \leq 0$. Thus,

$$\left[B_2^T P_3(j\omega I - A_2)^{-1}B_2\right]^* + \left[B_2^T P_3(j\omega I - A_2)^{-1}B_2\right] = B_2^T(-j\omega I - A_2^T)^{-1}(-P_3 A_2 - A_2^T P_3)(j\omega I - A_2)^{-1}B_2 \geq 0$$

Therefore Eq. (A7) follows from Eq. (A10).

To complete the proof it is enough to show that

$$\frac{\underline{\sigma}[I + K_e G_e(j\omega)]}{1 + \bar{\sigma}\left[K_2(j\omega I - A_2)^{-1}B_2\right]} \leq \underline{\sigma}[I + K(j\omega)G(j\omega)] \leq \frac{\underline{\sigma}[I + K_e G_e(j\omega)]}{1 - \bar{\sigma}\left[K_2(j\omega I - A_2)^{-1}B_2\right]} \tag{A11}$$

By a short computation we have

$$I + K_e G_e(j\omega) = [I + K_2(j\omega I - A_2)^{-1}B_2][I + K(j\omega)G(j\omega)]$$

Using some properties of singular values, it follows that

$$\begin{aligned}&\underline{\sigma}[I+K(j\omega)G(j\omega)]\underline{\sigma}\left[I+K_2(j\omega I-A_2)^{-1}B_2\right]\\&\quad\leq\underline{\sigma}[I+K_eG_e(j\omega)]\\&\quad\leq\underline{\sigma}[I+K(j\omega)G(j\omega)]\bar{\sigma}\left[I+K_2(j\omega I-A_2)^{-1}B_2\right]\end{aligned}\qquad\text{(A12)}$$

Also, we can show that

$$\begin{aligned}&1-\bar{\sigma}\left[K_2(j\omega I-A_2)^{-1}B_2\right]\\&\quad\leq\underline{\sigma}\left[I+K_2(J\omega I-A_3)^{-1}B_2\right]\leq\bar{\sigma}\left[I+K_2(J\omega I-A_2)^{-1}B_2\right]\\&\quad\leq 1+\bar{\sigma}\left[K_2(j\omega I-A_2)^{-1}B_2\right]\end{aligned}\qquad\text{(A13)}$$

Inequalities (A12) and (A13) prove Eq. (A11), and hence the proof of Theorem 3 is completed.

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